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# The Wigner Functions for a Spin-1/2 Relativistic Particle in the Presence of Magnetic Field

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**Abstract** This paper provides a study of Wigner functions for a spin-$1/2$ relativistic particle in the presence of magnetic field. Since the Dirac equation is described as a matrix equation, it is necessary to describe the Wigner function as a matrix function in phase space. What's more, this function is then proved to satisfy the Dirac equation with $\star$-product. Finally, by solving the $\star$-product Dirac equation, the energy levels as well as the Wigner functions for a spin-$1/2$ relativistic particle in the presence of magnetic field are obtained.

**Keywords** Spin-$1/2$ relativistic particle · Wigner function · Dirac equation · Spinor space

In recent years, the method of quantization has enjoyed a wide popularity in virtually all areas of physics. In fact, the phase-space formulation of quantum mechanics has its roots in the Wigner's classic work in 1932 [1], where he introduced a quasi-probability distribution function from the derivation of quantum correction terms to the Boltzmann formula. Since the phase-space formulation offers a framework in which quantum phenomena can be described through the language of classical physics, which appeals naturally to one's intuition. Furthermore, Wigner's quasi-probability distribution function in phase-space is a special (Weyl) representation of the density matrix. It has been useful in describing quantum transport in quantum optics [2, 3], nuclear physics, decoherence, quantum computing, quantum chaos, semi-classical approximation [4–8], etc. It is also of importance in signal processing. Nevertheless, a remarkable aspect of its internal logic, pioneered by Moyal [9–13, 16–20] et al., has only emerged in the last quarter-century. In this logically complete and self-standing formulation, one needs not choose sides-coordinate or momentum space

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because it works in full phase-space, accommodating the uncertainty principles, and it offers unique insights into the classical limit of quantum theory. Take [14] for example, the Wigner function of an ensemble of helium atoms was ingeniously tested and the result is same as that in theory.

In this letter, we generalize quantization from phase space to spinor space. Particularly, we discuss the spin-$1/2$ relativistic particles. To be clear, let us first discuss the Wigner function in a non-spinor space. As is known that there are three logical self-consistent methods of quantization from classical mechanics to quantum mechanics. The first one is the standard one utilizing operators in Hilbert space, developed by Heisenberg, Schrödinger, Dirac, and others in the twenties of the past century. The second one is path integrals method conceived by Dirac and constructed by Feynman. The third one is the Moyal multiplication (or star product) regularization based on the Wigner function in phase space. It is known that in phase space with the $n$ degree of freedom the general form of a time-independent Wigner function can be described by

$$W(\vec{x},\vec{p},t)=\frac{1}{(2\pi\hbar)^n}\int_{-\infty}^{\infty}dy\,e^{-i\vec{y}\vec{p}}\left\langle\vec{x}-\frac{\vec{y}}{2}|\hat{\rho}|\vec{x}+\frac{\vec{y}}{2}\right\rangle, \tag{1}$$

while in stationary states, the Wigner function can be written as

$$W(\vec{x},\vec{p})=\frac{1}{(2\pi\hbar)^n}\int_{-\infty}^{\infty}dy\,\psi^*\left(\vec{x}-\frac{\vec{y}}{2}\right)e^{-i\vec{y}\vec{p}}\psi\left(\vec{x}+\frac{\vec{y}}{2}\right). \tag{2}$$

This is a special representation of the density matrix. According to (1), we can prove that a time-independent Wigner function has the following dynamic evolution equation

$$\frac{\partial W}{\partial t}=-\frac{\vec{p}}{m}\frac{\partial W}{\partial\vec{x}}+\frac{\partial V}{\partial\vec{x}}\frac{\partial W}{\partial\vec{p}} \tag{3}$$

where $V$ is potential energy. And (3) is similar to Liouville theorem in the classical mechanics. In fact, with Hamiltonian $H(\vec{x},\vec{p})$ this equation can also be written as [10]

$$\frac{\partial W}{\partial t}=\frac{H\star W-W\star H}{i\hbar} \tag{4}$$

where the $\star$-product is

$$\star\equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_p}+\overleftarrow{\partial_p}\overrightarrow{\partial_x})}. \tag{5}$$

Since the $\star$-product involves exponential operators, which will cause much difficulty in real calculation. However, $\hbar$ is a very small volume. Therefore, $\star$-product, as a series expansion, can be expressed as [10]

$$f(x,p)\star g(x,p)=f\left(x+\frac{i\hbar}{2}\overrightarrow{\partial_p},p-\frac{i\hbar}{2}\overrightarrow{\partial_x}\right)g(x,p) \tag{6}$$

or

$$f(x,p)\star g(x,p)=f(x,p)g\left(x-\frac{i\hbar}{2}\overleftarrow{\partial_p},p+\frac{i\hbar}{2}\overleftarrow{\partial_x}\right). \tag{7}$$

In this way, Wigner functions satisfy the $\star$-eigenvalue equations [10]

$$H(x,p)\star W(x,p)=H\left(x+\frac{i\hbar}{2}\overrightarrow{\partial_p},p-\frac{i\hbar}{2}\overrightarrow{\partial_x}\right)W(x,p)=EW(x,p) \tag{8}$$

and

$$H(x,p)\star W(x,p)=W(x,p)H\left(x-\frac{i\hbar}{2}\overleftarrow{\partial_p},p+\frac{i\hbar}{2}\overleftarrow{\partial_x}\right)=EW(x,p). \tag{9}$$

Here $E$ is the energy eigenvalue of $H\psi=E\psi$. Using (8) and (9) Wigner functions can be obtained.

We are now in a position to write down the Wigner function and the Dirac equation in phase space for a particle with spin-$\frac{1}{2}$. To begin with, let us first discuss the Wigner function for a particle with spin-0. In this case the particle has no spin, the wave function has only one component, which we denote by $\phi$, the wave equation is the well known Klein-Gordon equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\phi+\frac{m^2c^2}{\hbar^2}\phi=0 \tag{10}$$

and its non-relativistic approximation is the Schrödinger equation. With this equation and its complex conjugate equation one can obtain a continuity equation which the probability density and probability current can satisfy. To be properly relativistic, $\rho$ is not a scalar, but a time component of a 4-vector, whose space component is $\vec{j}$. Then $\rho$ is given by

$$\rho=\frac{i\hbar}{2m}\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right). \tag{11}$$

Evidently, the $\rho$ here, unlike its expression in the Schrödinger equation, is not positively definite. This is because the Klein-Gordon equation is a second-order differential equation. $\phi$ and $\frac{\partial\phi}{\partial t}$ can be fixed arbitrarily at a given time. Therefore, $\rho$ may take on a negative value and lose the meaning of probability density.

Different from the Klein-Gordon equation, the Dirac equation for a spin-$\frac{1}{2}$ particle is of first order differential equation. In stationary case, the equation is given by [15]

$$\left[c\vec{\alpha}\cdot\vec{p}+\beta mc^2+V(x)\right]\psi=E\psi \tag{12}$$

and the four-dimensional vector $j^\mu=\overline{\psi}\gamma^\mu\psi$ is conserved, $\partial_\mu j^\mu=0$. Thus, the probability density $j^0$ is therefore like this

$$\mathrm{tr}(\rho)=j^0=\overline{\psi}\gamma^0\psi=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2, \tag{13}$$

which is clearly positive. Hence, the Wigner function is meaningful, and the quantization in phase-space can be extended to a particle with spin-$\frac{1}{2}$ in spinor space.

Now, let's define the Wigner functions $W$ for a particle with spin-$\frac{1}{2}$ as

$$\begin{aligned}W&=\frac{1}{2\pi\hbar}\int dy\,e^{-ipy}\left\langle x-\frac{y}{2}|\rho|x+\frac{y}{2}\right\rangle\\&=\frac{1}{2\pi\hbar}\int dy\,e^{-ipy}\left\langle x-\frac{y}{2}|\psi\right\rangle(\gamma^0)^2\left\langle\psi|x+\frac{y}{2}\right\rangle\\&=\frac{1}{2\pi\hbar}\int dy\,e^{-ipy}\psi^\dagger\left(x-\frac{y}{2}\right)\psi\left(x+\frac{y}{2}\right)\end{aligned} \tag{14}$$

where the spinors $\psi^\dagger(x-\frac{y}{2})$ and $\psi(x+\frac{y}{2})$ each has four components. So, the Wigner function for a particle with spin-$\frac{1}{2}$ is generally a $4\times 4$ matrix function holding sixteen components,

$$(W_{ij}) = \begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix} \tag{15}$$

where

$$W_{ij} = \frac{1}{2\pi\hbar}\int dy\, e^{-ipy}\psi_i^*\left(x-\frac{y}{2}\right)\psi_j\left(x+\frac{y}{2}\right) \quad (i, j = 1, 2, 3, 4). \tag{16}$$

Usually we split up the spinor $\psi$ into two two-component spinors $\phi$ and $\chi$. Thus, the Wigner function reduces to a diagonally partitioned matrix. If an appropriate representation is chosen, the Wigner function can also be transformed into a diagonal matrix, and its elements should be

$$W_{ij} = \frac{\delta_{ij}}{2\pi\hbar}\int dy\, e^{-ipy}\psi_i^*\left(x-\frac{y}{2}\right)\psi_j\left(x+\frac{y}{2}\right) \quad (i, j = 1, 2, 3, 4). \tag{17}$$

Now, generalizing the quantization of Dirac equation to phase space. It can be proved that in the phase space with spin-$1/2$ the usual product should be replaced by the $\star$-product, and the Wigner function can be described as a $4\times 4$ matrix function. The proof is the following.

$$\begin{aligned} H\star W &= (c\vec{\alpha}\cdot\vec{p} + \beta mc^2 + V(x))\star W \\ &= \frac{1}{2\pi\hbar}\left(c\vec{\alpha}\cdot\left(\vec{p}-\frac{i\hbar}{2}\overrightarrow{\partial}_{\vec{x}}\right) + \beta mc^2 + V\left(\vec{x}+\frac{i\hbar}{2}\overrightarrow{\partial}_{\vec{p}}\right)\right) \\ &\quad\times \int dy\, \exp[-i\vec{y}\cdot\vec{p})]\psi^\dagger\left(x-\frac{y}{2}\right)\psi\left(x+\frac{y}{2}\right) \\ &= \frac{1}{2\pi\hbar}\int dy\left(c\vec{\alpha}\cdot\left(\vec{p}-\frac{i\hbar}{2}\overrightarrow{\partial}_{\vec{x}}\right) + \beta mc^2 + V\left(\vec{x}+\frac{\hbar}{2}\vec{y}\right)\right) \\ &\quad\times \exp(-i\vec{y}\vec{p})\psi^\dagger\left(x-\frac{y}{2}\right)\psi\left(x+\frac{y}{2}\right) \\ &= \frac{1}{2\pi\hbar}\int dy\, \exp(-i\vec{y}\vec{p})\left(c\vec{\alpha}\cdot\left(-i\overrightarrow{\partial}_{\vec{p}}-\frac{i\hbar}{2}\overrightarrow{\partial}_{\vec{x}}\right) + \beta mc^2\right. \\ &\quad\left.+ V\left(\vec{x}+\frac{\hbar}{2}\vec{y}\right)\right)\psi\left(x+\frac{y}{2}\right)\psi^\dagger\left(x-\frac{y}{2}\right) \\ &= \frac{1}{2\pi\hbar}\int dy\, \exp(-i\vec{y}\vec{p})E\psi\left(x+\frac{y}{2}\right)\psi^\dagger\left(x-\frac{y}{2}\right) = EW. \end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned} W\star H &= W\star(c\vec{\alpha}\cdot\vec{p} + \beta mc^2 + V(x)) \\ &= \frac{1}{2\pi\hbar}\left[\int dy\, \exp(-i\vec{y}\cdot\vec{p})\psi^\dagger\left(x-\frac{y}{2}\right)\psi\left(x+\frac{y}{2}\right)\right] \end{aligned}$$

$$\times\left(c\vec{\alpha}\cdot\left(\vec{p}+\frac{i\hbar}{2}\overleftarrow{\partial}_{\vec{x}}\right)+\beta mc^2+V\left(\vec{x}-\frac{i\hbar}{2}\overleftarrow{\partial}_{\vec{p}}\right)\right)$$
$$=\frac{1}{2\pi\hbar}\int dy\,\exp(-i\vec{y}\vec{p})\psi\left(x+\frac{y}{2}\right)E\psi^{\dagger}\left(x-\frac{y}{2}\right)=EW. \tag{19}$$

Obviously, the matrix Wigner functions in phase space obey the Dirac equation with the $\star$-product.

In the following we investigate the spin-$1/2$ relativistic particle in the presence of external magnetic field. For the stationary situation, the Dirac equation can be defined by

$$\left[c\vec{\alpha}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)+\beta mc^2\right]\psi=E\psi \tag{20}$$

where

$$\psi(\vec{r})=\begin{pmatrix}\psi_a(\vec{r})\\ \psi_b(\vec{r})\end{pmatrix},\qquad \vec{\alpha}=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix},\qquad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}. \tag{21}$$

Thus, the Wigner function has the following form

$$W=\begin{pmatrix}W^{(a)} & 0\\ 0 & W^{(b)}\end{pmatrix}=\begin{pmatrix}W^{(a)}_{11} & W^{(a)}_{12} & 0 & 0\\ W^{(a)}_{21} & W^{(a)}_{22} & 0 & 0\\ 0 & 0 & W^{(b)}_{11} & W^{(b)}_{12}\\ 0 & 0 & W^{(b)}_{21} & W^{(b)}_{22}\end{pmatrix}. \tag{22}$$

In phase space the Dirac $\star$-eigenvalue equation, which has the same eigenvalue as that in (20), can be written as

$$\left[c\vec{\alpha}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)+\beta mc^2\right]\star W=E\star W. \tag{23}$$

A straightforward calculation leads to the following two equations

$$c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\star W^{(b)}=(E-mc^2)W^{(a)} \tag{24}$$

and

$$c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\star W^{(a)}=(E+mc^2)W^{(b)}. \tag{25}$$

Inserting (24) into (23), we have

$$\left[c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\right]\star\left[c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\right]\star W^{(a)}=(E^2-m^2c^4)W^{(a)} \tag{26}$$

where $\vec{\sigma}$ is the Pauli matrix. Let the uniform magnetic filed along $z$ direction, thus the magnetic vector potentials can be chosen

$$\vec{A}=\frac{\vec{B}\times\vec{r}}{2},\quad \text{i.e.}\quad A_x=\frac{1}{2}By,\quad A_y=-\frac{1}{2}Bx. \tag{27}$$

By utilizing the Bopp's shift, (26) can be written as

$$\left[c\vec{\sigma}\cdot\left(\left(\vec{p}-\frac{i\hbar}{2}\overrightarrow{\partial}_{\vec{x}}\right)+\frac{e}{c}\vec{A}\right)\right]\left[c\vec{\sigma}\cdot\left(\left(\vec{p}-\frac{i\hbar}{2}\overrightarrow{\partial}_{\vec{x}}\right)+\frac{e}{c}\vec{A}\right)\right]W^{(a)}=(E^2-m^2c^4)W^{(a)}. \tag{28}$$

Now, let's define $\omega$ as

$$\omega=\frac{eB}{2mc}. \tag{29}$$

Then in the two dimensions case, after some simple algebra we have

$$\begin{aligned}&c^2\bigg\{(p_1^2+p_2^2)+m^2\omega^2(x_1^2+x_2^2)-\frac{\hbar^2}{4}m^2\omega^2(\partial_{p_1}^2+\partial_{p_2}^2)-\frac{\hbar^2}{4}(\partial_{x_1}^2+\partial_{x_2}^2)\\&\quad+2m\omega(x_1p_2-x_2p_1)+\frac{\hbar^2}{2}m\omega(\partial_{x_2}\partial_{p_1}-\partial_{x_1}\partial_{p_2})+2m\hbar\omega\sigma_z\bigg\}W^{(a)}\\&=(E^2-m^2c^4)W^{(a)}.\end{aligned} \tag{30}$$

If the eigenstate of the operator $\sigma_z$ is chosen, the Wigner function can be simplified as

$$W^a=\begin{pmatrix}W_{11}^{(a)} & 0\\ 0 & W_{22}^{(a)}\end{pmatrix}. \tag{31}$$

Thus, we deduce the matrix-equation that Wigner functions satisfy for a particle with spin-1/2 in a non-relativistic field as the following

$$\begin{aligned}&c^2\bigg\{(p_1^2+p_2^2)+m^2\omega^2(x_1^2+x_2^2)-\frac{\hbar^2}{4}m^2\omega^2(\partial_{p_1}^2+\partial_{p_2}^2)-\frac{\hbar^2}{4}(\partial_{x_1}^2+\partial_{x_2}^2)\\&\quad+2m\omega(x_1p_2-x_2p_1)+\frac{\hbar^2}{2}m\omega(\partial_{x_2}\partial_{p_1}-\partial_{x_1}\partial_{p_2})\bigg\}\begin{pmatrix}W_{11}^{(a)} & 0\\ 0 & W_{22}^{(a)}\end{pmatrix}\\&=\begin{pmatrix}\epsilon_1W_{11}^{(a)} & 0\\ 0 & \epsilon_2W_{22}^{(a)}\end{pmatrix}.\end{aligned} \tag{32}$$

Here $\epsilon_1=(E^2-m^2c^4-2m\hbar\omega), \epsilon_2=(E^2-m^2c^4+2m\hbar\omega)$. With further calculation we arrive at the following equations

$$\begin{aligned}&c^2\bigg\{(p_1^2+p_2^2)+m^2\omega^2(x_1^2+x_2^2)-\frac{\hbar^2}{4}m^2\omega^2(\partial_{p_1}^2+\partial_{p_2}^2)-\frac{\hbar^2}{4}(\partial_{x_1}^2+\partial_{x_2}^2)\\&\quad+2m\omega(x_1p_2-x_2p_1)+\frac{\hbar^2}{2}m\omega(\partial_{x_2}\partial_{p_1}-\partial_{x_1}\partial_{p_2})\bigg\}W_{11}^{(a)}=\epsilon_1W_{11}^{(a)}\end{aligned} \tag{33}$$

and

$$\begin{aligned}&c^2\bigg\{(p_1^2+p_2^2)+m^2\omega^2(x_1^2+x_2^2)-\frac{\hbar^2}{4}m^2\omega^2(\partial_{p_1}^2+\partial_{p_2}^2)-\frac{\hbar^2}{4}(\partial_{x_1}^2+\partial_{x_2}^2)\\&\quad+2m\omega(x_1p_2-x_2p_1)+\frac{\hbar^2}{2}m\omega(\partial_{x_2}\partial_{p_1}-\partial_{x_1}\partial_{p_2})\bigg\}W_{22}^{(a)}=\epsilon_2W_{22}^{(a)}.\end{aligned} \tag{34}$$

These equations are similar to the Landau problem and are equivalent to a two dimensional relativistic oscillator with additional spin-orbit terms.

Now, let's introduce four new variables $X_i$ $(i = 1, 2, 3, 4)$

$$\begin{aligned} X_1 &= \left(\sqrt{\frac{1}{2m\omega}}p_1 + \sqrt{\frac{m\omega}{2}}x_2\right), \qquad X_2 = \left(\sqrt{\frac{1}{2m\omega}}p_2 + \sqrt{\frac{m\omega}{2}}x_1\right), \\ X_3 &= \left(\sqrt{\frac{1}{2m\omega}}p_2 - \sqrt{\frac{m\omega}{2}}x_1\right), \qquad X_4 = \left(\sqrt{\frac{1}{2m\omega}}p_1 - \sqrt{\frac{m\omega}{2}}x_2\right). \end{aligned} \tag{35}$$

Inserting (35) into (33), we can derive

$$\begin{aligned} &c^2\left\{3m\omega(X_2^2 + X_4^2) - m\omega(X_1^2 + X_3^2) - \frac{3\hbar^2}{8}m\omega(\partial^2_{X_2} + \partial^2_{X_4}) + \frac{\hbar^2}{8}m\omega(\partial^2_{X_1} + \partial^2_{X_3})\right\}W_{11}^{(a)} \\ &\quad = \epsilon_1 W_{11}^{(a)}. \end{aligned} \tag{36}$$

With two more new variables $\xi$ and $\eta$

$$\xi := \frac{2}{\hbar}(X_1^2 + X_3^2), \qquad \eta := \frac{2}{\hbar}(X_2^2 + X_4^2), \tag{37}$$

(36) can be rewritten as follows

$$c^2 m\omega\left[6\left(\frac{\eta}{4} - \eta\partial_\eta^2 - \partial_\eta\right) - 2\left(\frac{\xi}{4} - \xi\partial_\xi^2 - \partial_\xi\right)\right]W_{11}^{(a)} = \epsilon_1 W_{11}^{(a)} \tag{38}$$

Separating variables, $W_{11}^{(a)}(\xi, \eta) = W_{11}^{(a)}(\xi)W_{11}^{(a)}(\eta)$, we have

$$c^2 m\omega\left[\frac{\xi}{4} - \xi\partial_\xi^2 - \partial_\xi - \epsilon^1\right]W_{11}^{(a)}(\xi) = 0, \tag{39}$$

and

$$c^2 m\omega\left[\frac{\eta}{4} - \eta\partial_\eta^2 - \partial_\eta - \epsilon^2\right]W_{11}^{(a)}(\eta) = 0, \tag{40}$$

where $\epsilon_1 = 6\epsilon^2 - 2\epsilon^1$. In this way, we can finally find the solutions for (39) and (40)

$$W_{11}^{(a)}(\xi)_k = \frac{(-1)^k}{\pi\hbar}e^{-\xi/2}L_k(\xi), \qquad \epsilon^1 = \left(k + \frac{1}{2}\right)c^2 m\hbar\omega, \quad k = 0, 1, \ldots \tag{41}$$

and

$$W_{11}^{(a)}(\eta)_n = \frac{(-1)^n}{\pi\hbar}e^{-\eta/2}L_n(\eta), \qquad \epsilon^2 = \left(n + \frac{1}{2}\right)c^2 m\hbar\omega, \quad n = 0, 1, \ldots. \tag{42}$$

Thus, we have

$$\begin{aligned} W_{11}^{(a)}(\xi, \eta)_{nk} &= \frac{(-1)^{k+n}}{(\pi\hbar)^2}e^{-(\xi+\eta)/2}L_k(\xi)L_n(\eta), \\ \epsilon_1 &= 6\left(n + \frac{1}{2}\right)c^2 m\hbar\omega - 2\left(k + \frac{1}{2}\right)c^2 m\hbar\omega. \end{aligned} \tag{43}$$

And corresponding energy levels of $W_{11}^{(a)}(\xi,\eta)_{nk}$ is

$$E_{nk}^2 = m^2c^4 + 2m\hbar\omega + 6\left(n+\frac{1}{2}\right)c^2m\hbar\omega - 2\left(k+\frac{1}{2}\right)c^2m\hbar\omega. \tag{44}$$

Similarly, we can also find the solutions for $W_{22}^{(a)}$

$$\begin{aligned} W_{22}^{(a)}(\mu,\nu)_{nk} &= \frac{(-1)^{k+n}}{(\pi\hbar)^2}e^{-(\mu+\nu)/2}L_k(\mu)L_n(\nu), \\ \epsilon_2 &= 6\left(n+\frac{1}{2}\right)c^2m\hbar\omega - 2\left(k+\frac{1}{2}\right)c^2m\hbar\omega. \end{aligned} \tag{45}$$

In this case the corresponding energy levels of $W_{22}^{(a)}(\mu,\nu)_{nk}$ is

$$E_{nk}^2 = m^2c^4 - 2m\hbar\omega + 6\left(n+\frac{1}{2}\right)c^2m\hbar\omega - 2\left(k+\frac{1}{2}\right)c^2m\hbar\omega. \tag{46}$$

These are the Wignar functions and Energy levels for the spin-$1/2$ relativistic charged particle in the presence of magnetic field.

In sumamry, by defining a matrix Wigner function for a spin-$1/2$ relativistic particle, the paper first proved that the Wigner function can be described as a $4\times 4$ matrix function which obeys the Dirac equation, and in the phase space this Dirac equation should be described as a matrix equation with a $\star$-product. As a result, by solving the $\star$-product Dirac equations in phase space, Energy levels as well as the Wigner functions for a spin-$1/2$ relativistic charged particle in a magnetic field are obtained. In addition, in recent years more and more attention has been paid to the non-commutative feature of the Wigner function. The related works on the subject will be reported in our forthcoming studies.

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